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PROBLEM OF CONFLICT CONTROL WITH HEREDITARY INFORMATION†

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For a dynamical system subject to uncontrollable interference the problem [1–8] of the control that will guarantee an optimum performance index is considered, where the latter is given as a functional of the realized motion. The investigation concerns the case in which information about the history of the motion must be used. A functional-theoretic treatment is presented, reducing the initial problem to the construction of the upper convex hulls of certain auxiliary functions [7, 9–12] in multidimensional spaces. On the other hand, a method for reducing the problem to constructions in a space of much lower dimension is developed. The method is demonstrated on problems with typical performance indices.

1. FORMULATION OF THE PROBLEM

Suppose a system is described by the equation

$$dx/dt = A(t)x + f(t,u,v),\quad 0 \leqslant t_*^0 \leqslant t \leqslant \vartheta \tag{1.1}$$

$$x \in R^n,\ u \in R^r,\ v \in R^s$$

where x is the phase vector, u is the control vector, v is the interference vector, t_*^0, ϑ are given instants of time, n, r and s are given natural numbers, $A(t)$ and $f(t, u, v)$ are matrix-valued and vector-valued functions, respectively, piecewise continuous in t, where $f(t, u, v)$ is jointly continuous in all its arguments in its intervals of continuity with respect to t (the discontinuity points with respect to t of $f(t, u, v)$ are independent of u and v), both functions are continuous from the right at their discontinuity points, and u and v obey the constraints

$$u \in P,\ v \in Q \tag{1.2}$$

where P and Q are given compact sets; the saddle-point condition in a small game is satisfied [1; 6, p. 79], that is, for any $m \in R^n$ and $t \in [t_*^0, \vartheta]$

$$\min_{u \in P} \max_{v \in Q} \langle m, f(t,u,v) \rangle = \max_{v \in Q} \min_{u \in P} \langle m, f(t,u,v) \rangle \tag{1.3}$$

where the symbol $\langle \cdot , \cdot \rangle$ denotes the scalar product.

The admissible realizations are Borel-measurable functions $u[[\cdot\,]\vartheta) = \{u[t] \in P,\ t_*^0 \leqslant t < \vartheta\}$ and $v[t_*^0[\,\cdot\,]\vartheta) = \{u[t] \in Q,\ t_*^0 \leqslant t < \vartheta\}$. These realizations, by (1.1), generate absolutely continuous motions $x[t_*^0[\cdot\,]\,\vartheta] = \{x[t], t_*^0 \leqslant t < \vartheta\}$ (the initial state $x[t_*^0]$ is given).

We define the performance index γ of the motion $x[t_*^0[\,\cdot\,]\vartheta]$ as the functional $\gamma(x[t_*^0[\,\cdot\,]\vartheta])$ with the following structure. Choose a natural number N, times $t^{[i]} \in [t_*^0, \vartheta]$, $t^{[i+1]} > t^{[i]}$, $i = 1, \ldots, N-1$, $t^{[N]} = \vartheta$, constant matrices $D^{[i]}$ of dimensions $p^{[i]} \times n$, $1 \leqslant p^{[i]} \leqslant n$, $i = 1, \ldots, N$. The sequence $\{D^{[1]}x[t^{[1]}], \ldots, D^{[N]}x[t^{[1]}]$ forms a p-vector, $p = p^{[1]} + \ldots + p^{[N]}$. Choose some norm $\mu(\,\cdot\,)$ in the space R^p of all such sequences. Now define

$$\gamma = \gamma(x[t_*^0[\cdot]\vartheta]) = \mu(\{D^{[1]}x[t^{[1]}], \ldots, D^{[N]}x[t^{[N]}]\}) \tag{1.4}$$

This performance index may be specified in advance or the functional is defined as an approximation for the initial index $\gamma_*(x[t_*^0[\,\cdot\,]\vartheta])$, which takes a continuum of values of $x[t]$ into account.

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The problem is to find a control (or interference) designed to minimize the index γ (1.4) (which itself is designed to maximize).

These problems are combined, according to [7], into a two-person antagonistic differential game (u is the first player's move and v that of the second player). For any initial history $x[t_*^0[\cdot]t_*]$ ($t_*^0 \leqslant t_* < \vartheta$) this game has a value $\rho^0(x[t_*^0[\cdot]t_*])$. The game has a saddle point consisting of optimum strategies $\{u^0(x[t_*^0[\cdot]t], \varepsilon), v^0(x[t_*^0[\cdot]t], \varepsilon\}$, where $x[t_*^0[\cdot]t] = \{x[\tau], t_*^0 \leqslant \tau \leqslant t$ is the history of the motion realized by the actual time t; $\varepsilon > 0$ is some parameter of accuracy [6, 7]. The motions are generated in a discrete-time scheme [6, 7]. The optimum strategies u^0 and v^0 are constructed as extremum strategies [7, p. 150] for the functional $\rho^0(\cdot)$.

Thus, in order to construct an optimum control and a counter-optimum interference, we need an effective way to calculate the value of the game, using each successive history $x[t_*^0[\cdot]t]$ as the initial history.

In many cases, in order to construct optimum controls it is sufficient to consider only some part of the history up to the actual time t. For example, if the functional γ (1.4) is positional [7, 8], it is sufficient to rely on the current position $[t, x(t)]$ only.

In the differential game under consideration, with condition (1.3) satisfied, the saddle point is reached with pure strategies. If condition (1.3) is not satisfied, a solution must be sought in the class of mixed strategies [7, 9]. But even then the auxiliary constructions that make up the main part of this paper need no significant alteration.

2. FUNCTIONAL-THEORETIC TREATMENT

Suppose that by the time $t \in [t_*^0, \vartheta]$ the history actually realized is $x[t_*^0[\cdot]t]$. We will use the term functional position corresponding to that history for the sequence $\{t, \hat{\mathbf{z}}[t]\}$, where

$$\hat{\mathbf{z}}[t] = (x[t], \hat{\mathbf{x}}[t]), \quad \hat{\mathbf{x}}[t] = \{\hat{x}^{[1]}[t], \ldots, \hat{x}^{[N]}[t]\} \tag{2.1}$$

$$\hat{x}^{[i]}[t] = \begin{cases} D^{[i]}x[t^{[i]}], & t^{[i]} \leqslant t \\ D^{[i]}X[t^{[i]}, t]x[t], & t < t^{[i]} \end{cases}$$

Here $X[\tau, t]$ is a fundamental matrix of solutions for the equation $dx/d\tau = A(\tau)x$.

The index γ (1.4) may now be written in the form $\gamma = \mu(\hat{\mathbf{x}}[\vartheta])$.

The evolution of the functional position $\{t, \hat{\mathbf{z}}[t]\} = \{t, (x[t], \hat{\mathbf{x}}[t])\}$ is described by Eqs (1.1) and

$$d\hat{\mathbf{x}}[t]/dt = \hat{\mathbf{f}}(t, u, v), \; t_*^0 \leqslant t \leqslant \vartheta \tag{2.2}$$

where

$$\hat{\mathbf{f}}(t,u,v) = \{\hat{f}^{[1]}(t,u,v), \ldots, \hat{f}^{[N]}(t,u,v)\} \tag{2.3}$$

$$\hat{f}^{[i]}(t,u,v) = \begin{cases} D^{[i]}X[t^{[i]}, t]f(t,u,v), & t < t^{[i]} \\ 0, & t^{[i]} \leqslant t \end{cases}$$

The saddle-point condition in a small game for $\hat{\mathbf{f}}(t, u, v)$ is satisfied due to (1.3). The initial state $\hat{\mathbf{z}}[t_*^0] = (x[t_*^0], \hat{\mathbf{x}}[t_*^0])$ for system (1.1), (2.2) is uniquely defined by the initial state $x[t_*^0]$ of system (1.1).

Let us define a performance index $\hat{\gamma}$ for the motion $\hat{\mathbf{z}}[t_*^0[\cdot]\vartheta] = \{\hat{\mathbf{z}}[t], t_*^0 \leqslant t \leqslant \vartheta\}$ of system (1.1), (2.2)

$$\hat{\gamma} = \hat{\gamma}(\hat{\mathbf{z}}[\vartheta]) = \mu(\hat{\mathbf{x}}[\vartheta]) \tag{2.4}$$

where $\mu(\cdot)$ is the norm of (1.4). The value of the index $\hat{\gamma}$ (2.4) is the same as that of γ (1.4).

Let us consider the differential game (1.1), (2.2)–(2.4) in the space of functional positions $\{t, \hat{\mathbf{z}}[t]\}$, but now with the terminal payoff γ (2.4). This game has the value $\hat{\rho}^0(t_*, \hat{\mathbf{z}}[t_*])$ and a saddle point $\{\hat{u}^0(t, \hat{\mathbf{z}}[t], \varepsilon), \hat{v}^0(t, \hat{\mathbf{z}}[t], \varepsilon)\}$, where $\hat{\mathbf{z}}[t_*]$ denotes the initial state of system (1.1), (2.2) and $\hat{\mathbf{z}}[t]$ its actual state. The optimum strategies $\hat{u}^0(t, \hat{\mathbf{z}}[t]), \varepsilon)$ are constructed as extremum strategies [6, pp. 210, 220] for the value function $\hat{\rho}^0(t, \hat{\mathbf{z}}[t])$.

It follows from (1.1)–(1.4) and (2.1)–(2.4) that the value $\hat{\rho}^0(t, \hat{\mathbf{z}}[t])$ of game (1.1), (2.2)–(2.4) is the same as the value $\rho^0(x[t_*^0[\cdot]t])$ of game (1.1)–(1.4), and strategies that are optimum for game (1.1), (2.2)–(2.4), given condition (2.1), will determine actions u and v in the same way as optimum strategies for game (1.1)–(1.4). This means that games (1.1)–(1.4) and (1.1), (2.2)–(2.4) are essentially equivalent. Therefore, the terminal constructions of [6, 9–12] are transformed in a natural manner into constructions for the initial game (1.1)–(1.4). It must merely be taken into consideration that, unlike a standard differential system, in the case of (1.1), (2.2), (2.3) the possible state $\hat{\mathbf{z}}[t]$ are not vectors with arbitrary components $x[t], \hat{x}^{[i]}[t], i = 1, \dots, N$ but only vectors whose components satisfy conditions (2.1).

3. CALCULATING THE VALUE OF THE GAME

Suppose that the realized history of motion of system (1.1) is $x[t_*^0[\cdot]t_*], t_*^0 \leq t_* < \vartheta$; according to (2.1), this history uniquely defines the functional position $\{t_*, \hat{\mathbf{z}}[t_*]\} = \{t_*, (x[t_*], \hat{\mathbf{x}}[t_*])\}$.

Following the method of stochastic programme synthesis [6, p. 380], we introduce the programme extremum. To that end we prescribe a partition

$$\Delta_k = \Delta_k\{\tau_j\} = \{\tau_j : \tau_1 = t_*,\ \tau_{j+1} > \tau_j,\ \tau_{j+1} - \tau_j \leq \delta_k,\ j = 1, \dots, k,\ \tau_{k+1} = \vartheta\} \tag{3.1}$$

of the time interval $[t_*, \vartheta]$ in which we include all the times $t^{[i]} \in [t_*, \vartheta], i = 1, \dots, N$, of (1.4) and all the points of discontinuity of the functions $A(t)$ and $f(t, u, v)$. Associated with the partition Δ_k (3.1) are jointly independent random variables (r.v.) $\{\xi_1, \dots, \xi_k\}$ uniformly distributed in the interval $0 \leq \xi_j \leq 1, j = 1, \dots, k$. The ordered set $\{\xi_1, \dots, \xi_k\}$ will be treated as an elementary event ω in a probability space $\{\Omega, B_*, P\}$, where $\Omega = \{\omega\}$ is the unit cube in k-space, B_* is a Borel σ-algebra for the cube, and $P = P(B)$ is a Lebesgue measure on the cube, $B \in B_*$.

Suppose

$$\mathbf{l}(\omega) = \{l^{[i]}(\omega) \in R^{p^{[i]}},\ i = 1, \dots N\},\ \omega \in \Omega$$

is a p-dimensional vector random variable defined on $\{\Omega, B_*, P\}$. The programme extremum $e(\cdot)$ is defined by

$$e(x[t_*^0[\cdot]t_*], \Delta_k) = \hat{e}(t_*, \hat{\mathbf{z}}[t_*], \Delta_k) =$$

$$= \sup_{\|\mathbf{l}(\cdot)\| \leq 1}\left[\left\langle \mathbf{l}_*, \hat{\mathbf{x}}[t_*]\right\rangle + M\left\{\sum_{j=1}^{k}\int_{\tau_j}^{\tau_{j+1}} \max_{v \in Q}\min_{u \in P}\left\langle \mathbf{l}^*(\tau_j, \omega),\ \hat{\mathbf{f}}(\tau, u, v)\right\rangle d\tau\right\}\right] \tag{3.2}$$

where

$$\|\mathbf{l}(\cdot)\| = \operatorname*{vraimax}_{\omega \in \Omega} \mu^*(\mathbf{l}(\omega)),\ \mathbf{l}_* = M\{\mathbf{l}(\omega)\}$$

$$\mathbf{l}^*(\tau_j, \omega) = \mathbf{l}^*(\tau_j, \xi_1, \dots, \xi_j) = M\{\mathbf{l}(\xi_1, \dots, \xi_k) | \xi_1, \dots, \xi_j\},\ j = 1, \dots, k$$

Here the symbol $\mu^*(\cdot)$ denotes the norm adjoint to the norm $\mu(\cdot)$ of (1.4). The symbol $M\{\cdot\}$ denotes the mathematical expectation, and $M\{\cdot|\cdot\}$ denotes the conditional mathematical expectation.

It follows from [6, p. 401] and the equivalence of games (1.1)–(1.4) and (1.1), (2.2)–(2.4) that

$$\lim_{k \to \infty, \delta_k \to 0} e(x[t_*^0[\cdot]t_*], \Delta_k) = \lim_{k \to \infty, \delta_k \to 0} \hat{e}(t_*, \hat{\mathbf{z}}[t_*], \Delta_k) =$$

$$= \hat{\rho}^0(t_*, \hat{\mathbf{z}}[t_*]) = \rho^0(x[t_*^0[\cdot]t_*])$$

By [10], the programme extremum $e(\cdot)$ (3.2) may be computed recursively by constructing upper convex hulls $\varphi_j(\mathbf{l})$ for suitable functions $\psi_j(\mathbf{l})$, but now with a deterministic argument $\mathbf{l} = \{l^{[i]} \in R^{p[i]}, i = 1, \dots, N\}$. These hulls must be constructed for each j in the domain $\mathbf{L} = \{\mathbf{l}: \mu^*(\mathbf{l}) \leq 1\}$ of the space $R^p, p = p^{[1]} + \dots + p^{[N]}$. Thus, we obtain

$$e(x[t_*^0[\cdot]t_*],\Delta_k)=\hat{e}(t_*,\hat{\mathbf{z}}[t_*],\Delta_k)=\max_{\mathbf{l}\in\mathbf{L}}\left[\left\langle \mathbf{l},\ \hat{\mathbf{x}}[t_*]\right\rangle+\varphi_1(\mathbf{l})\right]$$

We are interested in situations in which the number N, and hence also p, are large. Therefore, if it is not possible in some specific situation to find an effective way of constructing the above hulls, the computations become intractable even for relatively low dimensions n of the phase vector x.

It is essential that the computation of $e(\cdot)$ (3.2) in terms of upper convex hulls $\varphi_j(\mathbf{l})$ in domains $\mathbf{L}$ of high dimensions may be contracted under fairly general conditions to similar constructions in spaces of much lower dimensions. This is because equality (3.2) may be transformed into

$$e(x[t_*^0[\cdot]t_*],\Delta_k)=\sup_{\|l(\cdot)\|\leqslant 1}\left[\sum_{i=1}^{h(t_*)}\left\langle l_*^{[i]},D^{[i]}x[t^{[i]}]\right\rangle+\right.$$
$$\left.+\left\langle m_*,X[\vartheta,t_*]x[t_*]\right\rangle+M\left\{\sum_{j=1}^{k}\int_{\tau_j}^{\tau_{j+1}}\max_{v\in Q}\min_{u\in P}\left\langle m(\tau_j,\omega),X[\vartheta,\tau]f(\tau,u,v)\right\rangle d\tau\right\}\right] \tag{3.3}$$

where

$$h(t)=\max i,\ t^{[i]}\leqslant t,\ i=1,\dots,N$$

(if there is no i ($i = 1, \dots, N$) such that $t^{[i]} \leqslant t$, then $h(t) = 0$)

$$l_*^{[i]}=M\{l^{[i]}(\omega)\},\ i=1,\dots,h(t_*),\ m_*=M\left\{\sum_{i=h(t_*)+1}^{N}X^T[t^{[i]},\vartheta]D^{[i]^T}l^{[i]}(\omega)\right\} \tag{3.4}$$

$$m(\tau_j,\omega)=M\left\{\sum_{i=h(\tau_j)+1}^{N}X^T[t^{[i]},\vartheta]D^{[i]^T}l^{[i]}(\omega)|\xi_1,\dots,\xi_j\right\},\ j=1,\dots,k$$

(the superscript T denotes transposition; Eqs (2.1) and (2.3) are taken into consideration in (3.3)). This enables us to work not with functions $\psi_j(\mathbf{l})$ and $\varphi_j(\mathbf{l})$ of the multidimensional vector $\mathbf{l} = \{l^{[1]}, \dots, l^{[N]}\}$ but with suitable functions of the vector

$$m=\sum_{i=h(\tau_j)+1}^{N}X^T[t^{[i]},\vartheta]D^{[i]T}l^{[i]},\ m\in R^n$$

and of the vectors $l^{[i]}$, $i = 1, \dots, h(\tau_j)$ which involve only some of the components of $\mathbf{l}$. Moreover, in many typical cases it is sufficient to work only with functions of the vector m. True, certain additional parameters are then necessary. This general statement, formulated here in brief, will be explained later in relation to specified material.

4. POSITIONAL FUNCTIONALS

Let us consider differential games (1.1)–(1.4) with the following performance indices (1.4)

$$\gamma_{(1)}=\mu_{(1)}(\{D^{[1]}x[t^{[1]}],\dots,D^{[N]}x[t^{[N]}]\})=\sum_{i=1}^{N}\mu^{[i]}(D^{[i]}x[t^{[i]}]) \tag{4.1}$$

$$\gamma_{(2)}=\mu_{(2)}(\{D^{[1]}x[t^{[1]}],\dots,D^{[N]}x[t^{[N]}]\})=\max_{i=1,\dots,N}\{\mu^{[i]}(D^{[i]}x[t^{[i]}])\} \tag{4.2}$$

$$\gamma_{(3)}=\mu_{(3)}(\{D^{[1]}x[t^{[1]}],\dots,D^{[N]}x[t^{[N]}]\})=\left(\sum_{i=1}^{N}(\mu^{[i]}(D^{[i]}x[t^{[i]}]))^2\right)^{1/2} \tag{4.3}$$

where $\mu^{[i]}(\cdot)$ are certain norms in $R^{p[i]}$, $i = 1, \dots, N$.

The functionals $\gamma_{(1)}, \gamma_{(2)}, \gamma_{(3)}$ are positional [7, 8], so that a sufficient information image [7, pp. 20, 134] for optimum strategies in games (1.1)–(1.4) for (4.1), (4.2) and (4.3) will be the actual position $\{t, x(t)\}$.

Reduced procedures for constructing the functions $\psi_j(\cdot)$ and their convex hulls $\varphi_j(\cdot)$ in suitable domains G_j for cases with functionals $\gamma_{(1)}$ and $\gamma_{(2)}$ are described in detail and rigorously justified in [7]. Here we will exhibit a construction for the functional $\gamma_{(3)}$, following a general scheme of contraction. This contraction, on the one hand, allows for the differences between $\gamma_{(3)}$ and $\gamma_{(1)}$, $\gamma_{(2)}$; on the other hand, it retains the general features of the corresponding constructions.

Accordingly, we consider game (1.1)–(1.4) with the index $\gamma_{(3)}$ (4.3). The norm $\mu^*_{(3)}(\cdot)$ adjoint to $\mu_{(3)}(\cdot)$ is

$$\mu^*_{(3)}(\mathbf{l}) = \left(\sum_{i=1}^{N} (\mu^{[i]*}(l^{[i]}))^2\right)^{1/2}, \quad \mathbf{l} = \{l^{[i]} \in R^{p^{[i]}}, \ i = 1,\ldots,N\}$$

where $\mu^{[i]*}(\cdot)$, $i = 1, \ldots, N$ are the norms adjoint to $\mu^{[i]}(\cdot)$. Therefore, when the supremum (3.3) is constructed, the random variables $m(\tau_j, \omega)$ must obey constraints that depend on the scalars

$$\nu^2(\tau_j, \omega) = 1 - \sum_{i=1}^{h(\tau_j)} (\mu^{[i]*}(l^{[i]}(\omega)))^2$$

It turns out that here, as in [7] for $\gamma_{(1)}$ and $\gamma_{(2)}$, we can now change from random variables $l^{[i]}(\omega)$, $m(\tau_j, \omega)$ and $\nu(\tau_j, \omega)$ to deterministic variables $l^{[i]}$, m and ν. Put

$$\Delta\psi_j(t_*, m) = \int_{\tau_j}^{\tau_{j+1}} \max_{v \in Q} \min_{u \in P} \langle m, X[\vartheta, \tau] f(\tau, u, v)\rangle d\tau, \quad m \in R^n, \ j = 1,\ldots,k$$

We will now construct a sequence of domains $G_j^{(3)}(t_*)$ in the space R^{n+1} of pairs (m, ν) and a sequence of functions $\varphi_j^{(3)}(t_*, m, \nu)$, $(m, \nu) \in G_j^{(3)}(t_*)$, $j = k + 1, k, \ldots, 1$. The construction will be recursive with respect to the meshes of the partition $\Delta_k \{\tau_j\}$ (3.1).

For $j = k + 1$, define

$$G_{k+1}^{(3)}(t_*) = \{(m, \nu): 0 \le \nu \le 1, m = 0\}, \quad \varphi_{k+1}^{(3)}(t_*, m, \nu) = 0, \ (m, \nu) \in G_{k+1}^{(3)}(t_*)$$

We proceed by induction. Suppose that for $j + 1$ we have already constructed the domain $G_{j+1}^{(3)}(t_*)$ and the function $\varphi_{j+1}^{(3)}(t_*, m, \nu)$, $(m, \nu) \in G_{j+1}^{(3)}(t_*)$. We first construct the domain $G_j^{(3)}(t_*)$ and an auxiliary function $\varphi_{j+1}^{(3)\prime}(t_*, m, \nu)$, $(m, \nu) \in G_j^{(3)}(t_*)$. On changing from τ_{j+1} to τ_j there are two possibilities. In the first, we have $h(\tau_j) = h(\tau_{j+1})$, i.e. τ_{j+1} is not the same as any of the times $t^{[i]}$. We then define

$$G_j^{(3)}(t_*) = G_{j+1}^{(3)}(t_*), \quad \varphi_{j+1}^{(3)\prime}(t_*, m, \nu) = \varphi_{j+1}^{(3)}(t_*, m, \nu)$$

The second possibility is $h(\tau_j) = h(\tau_{j+1}) - 1$, that is, $\tau_{j+1} = t^{[h]}$, $h = h(\tau_j) + 1$, in which case we define

$$\begin{aligned} G_j^{(3)}(t_*) = \{&(m, \nu): 0 \le \nu \le 1, \ m = m_* + X^T[t^{[h]}, \vartheta] D^{[h]T} l, \ l \in R^{p^{[h]}}, \\ &(\mu^{[h]*}(l))^2 \le \nu^2 - \nu_*^2, \nu_* \le \nu, (m_*, \nu_*) \in G_{j+1}^{(3)}(t_*)\} \end{aligned} \qquad (4.4)$$

$$\varphi_{j+1}^{(3)\prime}(t_*, m, \nu) = \max_{m_*, \nu_*} \varphi_{j+1}^{(3)}(t_*, m_*, \nu_*), \quad (m, \nu) \in G_j^{(3)}(t_*)$$

where the maximum defining the auxiliary function $\varphi_{j+1}^{(3)}(\cdot)$ is taken over all possible pairs (m_*, ν_*) corresponding, according to (4.4), to the given pair $(m, \nu) \in G_j^{(3)}(t_*)$.

We now define

$$\psi_j^{(3)}(t_*, m, \nu) = \Delta\psi_j(t_*, m) + \varphi_{j+1}^{(3)\prime}(t_*, m, \nu), \quad (m, \nu) \in G_j^{(3)}(t_*)$$

$$\varphi_j^{(3)}(t_*, m, \nu) = \{\psi_j^{(3)}(t_*, \cdot, \nu)\}^*_G, \quad G = G_{j,\nu}^{(3)}(t_*), \ 0 \le \nu \le 1$$

where $G^{(3)}_{j,\nu}(t_*)$ is the section of the domain $G^{(3)}_j(t_*)$ by a hyperplane ν = const.

Here the symbol $\{\psi(t_*, \cdot, \nu)\}^*_G$ denotes the upper convex hull of the function $\psi(t_*, m, \nu)$ which is constructed by taking convex hulls with respect to m in the domain G, with all other arguments fixed.

By definition, the upper convex hull is the minimum function concave in m that majorizes the function $\psi(t_*, m, \nu)$, $m \in G$.

Continuing the induction up to $j = 1$, we construct a domain $G_1^{(3)}(t_*)$ and function $\varphi_1^{(3)}(t_*, m, \nu)$, $(m, \nu) \in G_1^{(3)}(t_*)$. Then the quantity

$$e_{(3)}(x[t_*^0[\cdot]t_*], \Delta_k) = \max_{(m,\nu)\in G_1^{(3)}(t_*)}\left[\left((1-\nu^2)\sum_{i=1}^{h(t_*)}(\mu^{[i]}(D^{[i]}x[t^{[i]}]))^2\right)^{1/2} + \right.$$
$$\left. +\left\langle m, X[\vartheta, t_*]x[t_*]\right\rangle + \varphi_1^{(3)}(t_*, m, \nu)\right]$$

will have the properties of u- and v-stability [6, 7]. We will not prove this here—the proof is similar to that given in [7] for games with indices $\gamma_{(1)}$ and $\gamma_{(2)}$. In addition, an outline of the proof of these properties will be given below, in Section 5, for a more difficult case.

As in [7], we conclude from these properties that $e_{(3)}(\cdot)$ approximates the value $\rho_0^{(3)}(x[t_*^0[\cdot]t_*])$ of game (1.1)–(1.3), (4.3). Thus, the problem reduces to constructing the convex hulls $\varphi_j^{(3)}(t_*, \cdot, \nu)$ of the functions $\varphi_j^{(3)}(t_*, \cdot, \nu)$ in the domains $G_{j,\nu}^{(3)}(t_*)$, $0 \leqslant \nu \leqslant 1$, where the latter are of the same dimensions as that of the phase vector x of system (1.1), which is independent of the number N of points $t^{[i]}$. We emphasize that here, as in many other cases, including games with indices $\gamma_{(1)}$ and $\gamma_{(2)}$, upper convex hulls are constructed only for functions of the variable m, for fixed values of $\nu \in [0, 1]$. This is because the domains $G_j^{(3)}(t_*)$, $j = k+1, \ldots, 1$, are homogeneous in (m, ν), that is, if

$$(m, \nu) \in G_j^{(3)}(t_*) \text{ then } (\eta m, \eta\nu) \in G_j^{(3)}(t_*), \; \eta \geqslant 0, \eta\nu \leqslant 1 \tag{4.5}$$

Hence we conclude that the functions $\varphi_j^{(3)}(t_*, m, \nu)$, $j = k+1, \ldots, 1$ will be homogeneous of degree one in the variables (m, ν). Therefore, the construction of upper convex hulls of the functions $\psi_j^{(3)}(t_*, m, \nu)$ in the domains $G_j^{(3)}(t_*)$ with respect to the pair (m, ν) leads to the very same functions $\varphi_j^{(3)}(t_*, m, \nu)$ that were constructed above by convex closure with respect to m only, in sections of $G_{j,\nu}^{(3)}(t_*)$ obtained by fixing $\nu \in [0, 1]$.

We have thus presented a construction of the function $\varphi_1^{(3)}(\cdot)$ which, by the foregoing, defines the value of the game (1.1)–(1.4) and optimum strategies for the typical index (4.3).

In what follows, working with specific data, we will show that, generally speaking, the construction of the functions $\varphi_j(\cdot)$ requires the application of convex closure with respect to all arguments of a space obtained by completing the space R^n of vectors m by adding auxiliary parameters (such as ν). This important fact is crucial for the present paper.

5. NON-POSITIONAL FUNCTIONAL

Let us consider the system described by the equation

$$dx/dt = A(t)x + B(t)u + C(t)v, \quad 0 \leqslant t_*^0 \leqslant t \leqslant \vartheta \tag{5.1}$$
$$x \in R^n, \quad u \in P \subset R^r, \quad v \in Q \subset R^s$$

where $A(t)$, $B(t)$ and $C(t)$ are piecewise continuous matrix functions; as before, P and Q are compact sets, and t_*^0, ϑ are fixed

Two partitions of the time interval $[t_*^0, \vartheta]$ are given

$$\Delta_{N_q}\{t_q^{[i_q]}\} = \{t_q^{[i_q]} : t_q^{[1]} \geqslant t_*^0, \; t_q^{[i_q+1]} > t_q^{[i_q]}, \; i_q = 1, \ldots, N_q - 1\} \tag{5.2}$$
$$q = 1, 2$$
$$t_1^{[i_1]} \neq t_2^{[i_2]}, \quad i_1 = 1, \ldots, N_1, \quad i_2 = 1, \ldots, N_2$$
$$\max\{t_1^{[N_1]}, t_2^{[N_2]}\} = \vartheta$$

The performance index of the motion of system (5.1) is

$$\gamma_{(4)} = \gamma_{(4)}(x[t_*^0[\cdot]\vartheta]) = \sum_{i_1=1}^{N_1}\mu_1^{[i_1]}(D_1^{[i_1]}x[t_1^{[i_1]}]) + \max_{i_2=1,\dots,N_2}\{\mu_2^{[i_2]}(D_2^{[i_2]}x[t_2^{[i_2]}])\} \tag{5.3}$$

where $D_q^{[i_q]}$ are known constant $p_q^{[i_q]} \times n$ matrices, $1 \leqslant p_q^{[i_q]} \leqslant n$; $\mu_q^{[i_q]}(\cdot)$ are certain norms, $i_q = 1, \dots, N_q$, and $q = 1, 2$.

The linearity of (5.1) with respect to u and v is not essential. A direct extension to the case of a system (5.1) non-linear in u and v is obtained, for example, by following the scheme of [6, 7].

The functional $\gamma_{(4)}$ (5.3) is an additive combination of the functionals $\gamma_{(1)}$ (4.1) and $\gamma_{(2)}$ (4.2) but, unlike $\gamma_{(1)}$, $\gamma_{(2)}$ and $\gamma_{(3)}$ (4.3), it is no longer positional. To construct optimum strategies in a game with index $\gamma_{(4)}$ it is essential to use information not only on the actual position $\{t, x[t]\}$ but also on the history of the motion $[t_*^0[\cdot]t]$. The case of game (1.1)–(1.4) with index (5.3) will serve us as specific data for a convenient demonstration that, in general, when the programme extremum $e(\cdot)$ (3.2), (3.3) is being computed, the construction of the functions $\varphi_j(\cdot)$ involves convex closure with respect to the set of all arguments, comprising m and the additional parameters, and defining suitable domains G_j (in this case the "completed vectors" are the pairs (m, ν)). The domains $G_j^{(4)}(t_*)$ arising here no longer possess homogeneity property (4.5).

The procedure for computing $e(\cdot)$ (3.2), (3.3) in this case is as follows (the procedure was only briefly indicated in [7]).

Suppose that the actually realized history of the motion of system (5.1) is $x[t_*^0[\cdot]t_*]$, $t_*^0 \leqslant t_* < \vartheta$ and that choice has been made of a partition

$$\Delta_k = \Delta_k\{\tau_j\} = \{\tau_j : \tau_1 = t_*, \tau_{j+1} > \tau_j, j = 1,\dots,k, \tau_{k+1} = \vartheta\} \tag{5.4}$$

of the interval $[t_*, \vartheta]$, including all the points of discontinuity of the matrix functions $A(t)$, $B(t)$ and $C(t)$ and all the points $t_q^{[i_q]} \in [t_*, \vartheta]$, $i_q = 1, \dots, N_q$, $q = 1, 2$ of (5.2).

The functions $\Delta\psi_j(\cdot)$ are defined by

$$\Delta\psi_j(t_*, m) = \int_{\tau_j}^{\tau_{j+1}} \max_{v\in Q}\min_{u\in P}\langle m, \; X[\vartheta,\tau](B(\tau)u + C(\tau)v\rangle d\tau \tag{5.5}$$

$$m \in R^n, j = 1,\dots,k$$

Let us construct the functions $\varphi_j^{(4)}(t_*, m, \nu)$, $(m, \nu) \in G_j^{(4)}(t_*)$, $m \in R^n$, $\nu \in R$, $j = k+1, k, \dots, 1$. If $j = k+1$, we define

$$G_{k+1}^{(4)}(t_*) = \{(m,\nu):\; m = 0,\; 0 \leqslant \nu \leqslant 1\},\; \varphi_{k+1}^{(4)}(t_*, m, \nu) = 0,\; (m,\nu) \in G_{k+1}^{(4)}(t_*) \tag{5.6}$$

Suppose that for $1 < j+1 \leqslant k+1$ domains $G_{j+1}^{(4)}(t_*)$ and functions $\varphi_{j+1}^{(4)}(t_*, m, \nu)$ have already been constructed. Put

$$h_q(\tau) = \max i_q,\; t_q^{[i_q]} \leqslant \tau,\; i_q = 1,\dots,N_q \tag{5.7}$$

(if there is no i_q such that $t_q^{[i_q]} \leqslant \tau$, then $h_q(\tau) = 0$, $q = 1, 2$).

The partition Δ_k is chosen in such a way that for any $j = 1, \dots, k$ only one of the following three possibilities can occur

1. $h_1(\tau_{j+1}) = h_1(\tau_j)$, $h_2(\tau_{j+1}) = h_2(\tau_j)$, that is, the time τ_{j+1} is not a point of the partition $\Delta_{Nq}\{t_q^{[i_q]}\}$ (5.2), $q = 1, 2$;
2. $h_1(\tau_{j+1}) = h_1(\tau_j) + 1$, $h_2(\tau_{j+1}) = h_2(\tau_j)$, that is, the time τ_{j+1} is a point $t_1^{[h1(\tau_{j+1})]}$ of the partition $\Delta_{N_1}\{t_1^{[i_1]}\}$;
3. $h_1(\tau_{j+1}) = h_1(\tau_j) + 1$, $h_2(\tau_{j+1}) = h_2(\tau_j) + 1$, that is, the time τ_{j+1} is a point $t_2^{[h2(\tau_{j+1})]}$ of $\Delta_{N_2}\{t_2^{[i_2]}\}$;

We will first construct the domain $G_j^{(4)}(t_*)$ and the auxiliary function $\varphi_{j+1}^{(4)}(t_*, m, \nu)$, $(m, \nu) \in G_j^{(4)}(t_*)$. In case 1 we put

$$G_j^{(4)}(t_*) = G_{j+1}^{(4)}(t_*),\; \varphi_{j+1'}^{(4)}(t_*, m, \nu) = \varphi_{j+1}^{(4)}(t_*, m, \nu),\; (m,\nu) \in G_j^{(4)}(t_*) \tag{5.8}$$

In case 2 we define

$$G_j^{(4)}(t_*) = \{(m,\nu):\ m = m_* + X^T[t_1^{[h]},\vartheta]D_1^{[h]T}l,\ l \in R^{p_1^{[h]}},$$
$$\mu_1^{[h]*}(l) \leq 1,\ h = h_1(\tau_j)+1,\ (m_*,\nu) \in G_{j+1}^{(4)}(t_*)\} \tag{5.9}$$

where $\mu_q^{[i_q]*}(\cdot)$ are the norms adjoint to the norms $\mu_q^{[i_q]}(\cdot)$ of (5.3), $i_q = 1,\ldots,N_q$, $1 = 1, 2$. The function $\varphi_{j+1}^{(4)}(\cdot)$ is then constructed as follows:

$$\varphi_{j+1}^{(4)\prime}(t_*,m,\nu) = \max_{m_*} \varphi_{j+1}^{(4)}(t_*,m_*,\nu),\ (m,\nu) \in G_j^{(4)}(t_*) \tag{5.10}$$

The maximum in (5.10) is calculated over all vectors m_* corresponding to the given pair $(m, \nu) \in G_j^{(4)}(t_*)$ according to (5.9).

In case we define

$$G_j^{(4)}(t_*) = \{(m,\nu):\ 0 \leq \nu \leq 1,\ m = m_* + X^T[t_2^{[h]},\vartheta]D_2^{[h]T}l,\ l \in R^{p_2^{[h]}},$$
$$\mu_2^{[h]*}(l) \leq \nu - \nu_*,\ \nu_* \leq \nu,\ h = h_2(\tau_j)+1,\ (m_*,\nu_*) \in G_{j+1}^{(4)}(t_*)\} \tag{5.11}$$

$$\varphi_{j+1}^{(4)\prime}(t_*,m,\nu) = \max_{m_*,\nu_*} \varphi_{j+1}^{(4)}(t_*,m_*,\nu_*),\ (m,\nu) \in G_j^{(4)}(t_*) \tag{5.12}$$

The maximum in (5.12) is calculated over all pairs (m_*, ν_*) corresponding to the given pair $(m, \nu) \in G_j^{(4)}(t_*)$ according to (5.11).

We now define

$$\varphi_j^{(4)}(t_*,m,\nu) = \{\psi_j^{(4)}(t_*,\cdot,\cdot)\}_G^*,\ \ G = G_j^{(4)}(t_*) \tag{5.13}$$
$$\psi_j^{(4)}(t_*,m,\nu) = \Delta\psi_j(t_*,m) + \varphi_{j+1}^{(4)\prime}(t_*,m,\nu),\ (m,\nu) \in G_j^{(4)}(t_*)$$

The symbol $\{\psi(t_*, \cdot, \cdot)\}$ in (5.13) denotes the upper convex hull of the function $\psi(t_*, m, \nu)$; it is constructed by convex closure with respect to the combined argument (m, ν) in the domain G.

Continuing the induction up to $j = 1$, we obtain a domain $G_1^{(4)}(t_*)$ and a function $\varphi_1^{(4)}(t_*, m, \nu)$, $(m, \nu) \in G_1^{(4)}(t_*)$.

The domains $G_j^{(4)}(t_*)$ will be convex compact subsets of R^{n+1}, and the functions $\varphi_j^{(4)}(t_*, m, \nu)$ and $\varphi_{j+1}^{(4)}(t_*, m, \nu)$ will be concave, bounded and at least upper semicontinuous [13, p. 51] on $G_j^{(4)}(t_*)$, $j = 1, \ldots, k$.

Put

$$\sigma(x[t_*^0[\cdot]t_*]) = \sum_{i_1=1}^{h_1(t_*)} \mu_1^{[i_1]}(D_1^{[i_1]}x[t_1^{[i_1]}]) \tag{5.14}$$
$$\varkappa(x[t_*^0[\cdot]t_*]) = \max_{i_2=1,\ldots,h_2(t_*)} \{\mu_2^{[i_2]}(D_2^{[i_2]}x[t_2^{[i_2]}])\}$$

Define

$$e_{(4)}(x[t_*^0[\cdot]t_*],\Delta_k) = \sigma(x[t_*^0[\cdot]t_*]) + \max_{(m,\nu)\in G_1^{(4)}(t_*)} [\varkappa(x[t_*^0[\cdot]t_*])(1-\nu) +$$
$$+ \langle m,\ X[\vartheta,t_*]x[t_*]\rangle + \varphi_1^{(4)}(t_*,m,\nu)] \tag{5.15}$$

We will show that the quantity $e_{(4)}(\cdot)$ (5.15) has the important properties of u- and ν-stability [6, pp. 208, 216; 7].

Theorem 5.1. (u-stability of $e_{(4)}(\cdot)$.) Suppose the history of the motion of system (5.1) that has been realized is $x[t_*^0[\cdot]t_*]$, $t^0_* \leq t_* < \vartheta$, and that a partition $\Delta_k[\tau_j]$ (5.4) of the time interval $[t_*, \vartheta]$ has been chosen. Then, for any admissible realization $\nu_*[t_*[\cdot]t^*) = \{\nu_*[\tau] \in Q, t_* \leq \tau < t^*\}$, where $t^* = \tau_2 \in \Delta_k[\tau_j]$, an admissible realization $u[t_*[\cdot]t^*) = \{u[\tau] \in P, t_* \leq \tau < t^*\}$ exists, such that, under the action of these controls, the realized history $x[t^0_*[\cdot]t^*]$ is such that

$$e_{(4)}(x[t^0_*[\cdot]t^*],\Delta^*_{k^*})-e_{(4)}(x[t^0_*[\cdot]t_*],\Delta_k)\leqslant 0$$

where $\Delta^*_{k^*}\{\tau^*_j\}$ is the partition of the interval $[t^*, \vartheta]$, $\tau^*_j = \tau_{j+1} \in \Delta_k, j = 1, \ldots, k^*, k^* = k-1, \tau^*_{k^*+1} = \vartheta$, generated by Δ_k.

Proof. By constructions (5.5)–(5.13), the relationship between the partitions $\Delta_k\{\tau_j\}$ and $\Delta^*_{k^*}\{\tau^*\}$, we obtain identities

$$G_1^{(4)}(t^*)\equiv G_2^{(4)}(t_*),\quad \varphi_1^{(4)}(t^*,m,\nu)\equiv\varphi_2^{(4)}(t_*,m,\nu) \tag{5.16}$$

We now define an auxiliary functional on the possible realizations $x[t^0_*[\cdot]t^*]$, besides the quantity $e_{(4)}(x[t^0_*[\cdot]t^*],\Delta^*_{k^*})$ (5.15)

$$e'_{(4)}(x[t^0_*[\cdot]t_*],t^*,x[t^*],\Delta^*_{k^*})=\sigma(x[t^0_*[\cdot]t_*])+\max_{(m,\nu)\in G_1^{(4)}(t_*)}[\varkappa(x[t^0_*[\cdot]t_*])(1-\nu)+$$
$$+\langle m, X[\vartheta,t^*]x[t^*]\rangle+\varphi_2^{(4)\prime}(t_*,m,\nu)] \tag{5.17}$$

It can be shown that

$$e'_{(4)}(x[t^0_*[\cdot]t_*],t^*,x[t^*],\Delta^*_{k^*})=e_{(4)}(x[t^0_*[\cdot]t^*],\Delta^*_{k^*}) \tag{5.18}$$

whatever history $x[t^0_*[\cdot]t^*]$ is realized.

Indeed, three cases are possible.

In the first case, we have $h_1(t^*) = h_1(t_*), h_2(t^*) = h_2(t_*)$. It then follows from (5.7) and (5.14) that $\sigma(x[t^0_*[\cdot]t^*]) = \sigma(x[t^0_*[\cdot]t_*]), \varkappa(x[t^0_*[\cdot]t^*])$. Thus, comparing (5.15)–(5.17) and taking note of (5.8) ($j = 1$), we obtain the required equality (5.18).

In the second case, we have $h_1(t^*) = h_1(t_*)+1, h_2(t^*) = h_2(t_*)$. Now, by (5.7) and (5.14), we have

$$t^*=t_1^{[h]},\ \sigma(x[t^0_*[\cdot]t^*])=\sigma(x[t^0_*[\cdot]t_*])+\mu_1^{[h]}(D_1^{[h]}x[t^*])$$
$$\varkappa(x[t^0_*[\cdot]t^*])=\varkappa(x[t^0_*[\cdot]t_*]),\quad h=h_1(t^*)$$

Then, by (5.15) and (5.16), a pair $(m^0_*, \nu) \in G_2^{(4)}(t_*)$ exists such that

$$e_{(4)}(x[t^0_*[\cdot]t^*],\Delta^*_{k^*})=\sigma(x[t^0_*[\cdot]t_*])+\mu_1^{[h]}(D_1^{[h]}x[t^*])+$$
$$+\varkappa(x[t^0_*[\cdot]t_*])(1-\nu^0)+\langle m^0_*,\ X[\vartheta,t^*]x[t^*]\rangle+\varphi_2^{(4)}(t_*,m^0_*,\nu^0),\quad h=h_1(t^*) \tag{5.19}$$

We now define vectors $m^0 \in R^n$ and $l^0 \in R^{p1[h]}$ from the conditions

$$\langle l^0,\ D_1^{[h]}x[t^*]\rangle=\max_{\mu_1^{[h]*}(l)\leqslant 1}\langle l,\ D_1^{[h]}x[t^*]\rangle=\mu_1^{[h]}(D_1^{[h]}x[t^*])$$
$$m^0=m^0_*+X^T[t^*,\vartheta]D_1^{[h]T}l^0 \tag{5.20}$$

We have $(m^0, \nu^0) \in G_1^{(4)}(t_*)$ (see (5.9)). It follows from (5.10) ($j = 1$) and (5.20) that $\varphi_2^{(4)}(t_*, m^0, \nu^0) \geqslant \varphi_2^{(4)}(t_*, m^0, \nu^0)$. Consequently, by (5.17), taking (5.19) and (5.20) into consideration, we have

$$e'_{(4)}(x[t^0_*[\cdot]t_*],t^*,x[t^*],\Delta^*_{k^*})\geqslant\varkappa(x[t^0_*[\cdot]t_*])(1-\nu^0)+\mu_1^{[h]}(D_1^{[h]}x[t^*])+$$
$$+\sigma(x[t^0_*[\cdot]t_*])+\langle m^0_*,X[\vartheta,t^*]x[t^*]\rangle+\varphi_2^{(4)}(t_*,m^0_*,\nu^0)=e_{(4)}(x[t^0_*[\cdot]t^*],\Delta^*_{k^*}) \tag{5.21}$$

On the other hand, in the present case, it follows from the construction of the domain $G_1^{(4)}(t_*)$ (5.9) and the function $\varphi_2^{(4)}(t_*, m, \nu)$ (5.10) that, for every pair $(m, \nu) \in G_1^{(4)}(t_*)$, vectors $m_*(m, \nu) \in R^n$ and $l(m, \nu) \in R^{p[h]1}$, $h = h_1(t^*)$ exist such that

$$(m_*(m,\nu),\nu)\in G_2^{(4)}(t_*),\ \ \mu_1^{[h]*}(l(m,\nu))\leqslant 1,\ \ m_*(m,\nu)+X^T[t^*,\vartheta]D_1^{[h]T}l(m,\nu)=m$$
$$\varphi_2^{(4)\prime}(t_*,m,\nu)=\varphi_2^{(4)}(t_*,m_*(m,\nu),\nu)$$

Then, by (5.17), a pair $(m_0, \nu_0) \in G_1^{(4)}(t_*)$ exists, for which

$$e'_{(4)}(x[t_*^0[\cdot]t_*], t^*, x[t^*], \Delta_{k^*}^*) = \sigma(x[t_*^0[\cdot]t_*]) + \varkappa(x[t_*^0[\cdot]t_*])(1-\nu_0) +$$
$$+\langle m_0,\ X[\vartheta, t^*]x[t^*]\rangle + \varphi_2^{(4)'}(t_*, m_0, \nu_0) \tag{5.22}$$

For the vectors $m_{*0} = m_*(m_0, \nu_0)$ and $l_0(m_0, \nu_0)$ corresponding to this pair (m_0, ν_0), the following relations hold

$$(m_{*0}, \nu_0) \in G_2^{(4)}(t_*),\ \ \mu_1^{[h]*}(l_0) \leqslant 1,\ \ m_{*0} + X^T[t^*, \vartheta]D_1^{[h]T} l_0 = m_0 \tag{5.23}$$
$$\varphi_2^{(4)'}(t_*, m_0, \nu_0) = \varphi_2^{(4)}(t_*, m_{*0}, \nu_0)$$

In view of (5.16), (5.22) and (5.23), it now follows from (5.15) that

$$e_{(4)}(x[t_*^0[\cdot]t^*], \Delta_{k^*}^*) \geqslant \sigma(x[t_*^0[\cdot]t_*]) + \varkappa(x[t_*^0[\cdot]t_*])(1-\nu_0) + \langle l_0, D_1^{[h]}x[t^*]\rangle +$$
$$+\langle m_{*0}, X[\vartheta, t^*]x[t^*]\rangle + \varphi_2^{(4)'}(t_*, m_0, \nu_0) = e'_{(4)}(x[t_*^0[\cdot]t_*], t^*, x[t^*], \Delta_{k^*}^*) \tag{5.24}$$

Inequalities (5.21) and (5.24) prove equality (5.18) in the second case.

In the third case we have $h_1(t^*) = h_1(t_*)$, $h_2(t^*) = h_2(t_*) + 1$. In that case the time t^* is a point $t_2^{[h]}$ of the partition $\Delta_{N2}\{t_2^{[i2]}\}$, $h = h_2(t^*)$. According to (5.15), in view of (5.14) and (5.16), we see that a pair $(m_*^0, \nu_*^0) \in G_2^{(4)}(t_*)$ exists such that

$$e_{(4)}(x[t_*^0[\cdot]t^*], \Delta_{k^*}^*) = \sigma(x[t_*^0[\cdot]t_*]) + \langle m_*^0, X[\vartheta, t^*]x[t^*]\rangle +$$
$$+\varphi_2^{(4)}(t_*, m_*^0, \nu_*^0) + \max\{\varkappa(x[t_*^0[\cdot]t_*]), \mu_2^{[h]}(D_2^{[h]}x[t^*])\}(1-\nu_*^0) \tag{5.25}$$

Define a pair $(m^0, \nu^0) \in G_1^{(4)}(t_*)$ and vector $l^0 \in R^{p[h]2}$ by the conditions

$$\varkappa(x[t_*^0[\cdot]t_*])(1-\nu^0) + \mu_2^{[h]}(D_2^{[h]}x[t^*])(\nu^0 - \nu_*^0) =$$
$$= \max_{\nu_*^0 \leqslant \nu \leqslant 1} [\varkappa(x[t_*^0[\cdot]t_*])(1-\nu) + \mu_2^{[h]}(D_2^{[h]}x[t^*])(\nu - \nu_*^0)] =$$
$$= \max\{\varkappa(x[t_*^0[\cdot]t_*]),\ \mu_2^{[h]}(D_2^{[h]}x[t^*])\}(1-\nu_*^0) \tag{5.26}$$
$$\langle l^0, D_2^{[h]}x[t^*]\rangle = \max_{\mu_2^{[h]*}(l) \leqslant \nu^0 - \nu_*^0} \langle l, D_2^{[h]}x[t^*]\rangle = \mu_2^{[h]}(D_2^{[h]}x[t^*])(\nu^0 - \nu_*^0)$$
$$m^0 = m_*^0 + X^T[t^*, \vartheta]D_2^{[h]T} l^0$$

It then follows from (5.17), in view of (5.12), (5.25) and (5.26), that

$$e'_{(4)}(x[t_*^0[\cdot]t_*], t^*, x[t^*], \Delta_{k^*}^*) \geqslant \varkappa(x[t_*^0[\cdot]t_*])(1-\nu^0) + \mu_2^{[h]}(D_2^{[h]}x[t^*])(\nu^0 - \nu_*^0) +$$
$$+\sigma(x[t_*^0[\cdot]t_*]) + \langle m_*^0, X[\vartheta, t^*]x[t^*]\rangle + \varphi_2^{(4)}(t_*, m_*^0, \nu_*^0) = e_{(4)}(x[t_*^0[\cdot]t^*]), \Delta_{k^*}^*) \tag{5.27}$$

In addition, in the third case, by (5.17), a pair $(m^0, \nu^0) \in G_1^{(4)}(t_*)$ exists such that

$$e'_{(4)}(x[t_*^0[\cdot]t_*], t^*, x[t^*], \Delta_{k^*}^*) = \sigma(x[t_*^0[\cdot]t_*]) + \varkappa(x[t_*^0[\cdot]t_*])(1-\nu_0) +$$
$$+\langle m_0, X[\vartheta, t^*]x[t^*]\rangle + \varphi_2^{(4)'}(t_*, m_0, \nu_0) \tag{5.28}$$

And then, by the construction of the domain $G_1^{(4)}(t_*)$ (5.11) and the function $\varphi_2^{(4)}(t_*, m, \nu)$ (5.12), a pair $(m_{*0}, \nu_{*0}) \in G_2^{(4)}(t_*)$ and a vector $l_0 \in R^{p[h]2}$, $h = h_2(t^*)$ satisfying the conditions

$$0 \leqslant \nu_{*0} \leqslant \nu_0,\ \mu_2^{[h]*}(l_0) \leqslant \nu_0 - \nu_{*0},\ m_{*0} + X^T[t^*, \vartheta]D_2^{[h]T} l_0 = m_0$$
$$\varphi_2^{(4)'}(t_*, m_0, \nu_0) = \varphi_2^{(4)}(t_*, m_{*0}, \nu_{*0}) \tag{5.29}$$

exist. Here the pair (m_0, ν_0) of (5.28) and the sequence $\{(m_{*0}, \nu_{*0}), l_0\}$ of (5.29) are defined in the same order in which previously, in the second case, we defined the pair (m_0, ν_0) of (5.22) and the vectors m_{*0}, l_0 of (5.23). The only difference is that, instead of the vector $m_*(m, \nu)$, which was defined in the second case by solving the problem of the maximum of (5.10) as a function of the pair (m, ν), solution of the problem of the maximum of (5.12) now yields a pair (m_*, ν_*) as a function of the pair (m, ν).

We now deduce from (5.15), in accordance with (5.16), (5.28) and (5.29), that

$$e_{(4)}(x[t_*^0[\cdot]t^*],\Delta^*_{k^*}) \geq \sigma(x[t_*^0[\cdot]t_*]) + \varkappa(x[t_*^0[\cdot]t_*])(1-\nu_0) + \langle l_0, D_2^{[h]}x[t^*]\rangle +$$
$$+\langle m_{*0}, X[\vartheta,t^*]x[t^*]\rangle + \varphi_2^{(4)'}(t_*,m_0,\nu_0) = e'_{(4)}(x[t_*^0[\cdot]t_*],t^*,x[t^*],\Delta^*_{k^*}) \qquad (5.30)$$

The proof of this inequality relies on the chain of inequalities

$$\max\{\varkappa(x[t_*^0[\cdot]t_*]),\mu_2^{[h]}(D_2^{[h]}x[t^*])\}(1-\nu_{*0}) \geq$$
$$\geq \varkappa(x[t_*^0[\cdot]t_*])(1-\nu_0) + \mu_2^{[h]}(D_2^{[h]}x[t^*])(\nu_0-\nu_{*0}) \geq$$
$$\geq \varkappa(x[t_*^0[\cdot]t_*])(1-\nu_0) + \langle l_0, D_2^{[h]}x[t^*]\rangle$$

Inequalities (5.27) and (5.30) prove equality (5.18) in the last, third case.

We now consider $W = W(t^*;\, t_*,\, x[t_*],\, v_*[t_*[\,\cdot\,]t^*)) \subset R^n$—the domain of attainability by the time t^* for motions of system (5.1) generated from the position $\{t_*, x[t_*]\}$ by any admissible control $u[t_*[\,\cdot\,]t^*)$ paired with $v_*[t_*[\,\cdot\,]t^*)$. This set is a non-empty, convex, compact subset of R^n. Since the functions $\varphi_j^{(4)}(t_*, m, \nu)$ are concave jointly in (m, ν), it follows, reasoning according to the scheme of [2, p. 120; 6, p. 320], which makes use of the Kakutani fixed-point theorem [14, p. 638], that we can verify the existence of a pair $\{(m^0, \nu^0), x^0\} \in E = [G_1^{(4)}(t_*) \times W]$ satisfying the following two conditions simultaneously

$$\varkappa(x[t_*^0[\cdot]t_*])(1-\nu^0) + \langle m^0, X[\vartheta,t^*]x^0\rangle + \varphi_2^{(4)'}(t_*,m^0,\nu^0) =$$
$$= \max_{(m,\nu)\in G_1^{(4)}(t_*)} [\text{Idem}(m^0 \to m, \nu^0 \to \nu)] \qquad (5.31)$$

$$\langle m^0, X[\vartheta,t^*]x^0\rangle = \min_{x\in W}\langle m^0, X[\vartheta,t^*]x\rangle \qquad (5.32)$$

The abbreviation "Idem" on the right of the equality denotes the expression obtained from the expression on the left of the equality by making the substitution shown in brackets.

Let $u^0[t_*[\,\cdot\,]t^*)$ be an admissible control which, paired with $v_*[t_*[\,\cdot\,]t^*)$, takes the motion of the system to a point $x^0 \in W$. When that is done the realized history of the motion is $x^0[t_*^0[\,\cdot\,]t^*]$.

Then, using the Cauchy formula and the legitimacy of performing the minimization operation under the integral sign, we deduce from (5.32) that

$$\int_{t_*}^{t^*} \langle m^0,\ X[\vartheta,\tau]B(\tau)u^0[\tau]\rangle d\tau = \int_{t_*}^{t^*} \min_{u\in P}\langle m^0,\ X[\vartheta,\tau]B(\tau)u\rangle d\tau \qquad (5.33)$$

and by (5.17), noting (5.31), we obtain

$$e'_{(4)}(x^0[t_*^0[\cdot]t_*],t^*,x^0,\Delta^*_{k^*}) = \varkappa(x[t_*^0[\cdot]t_*])(1-\nu^0) + \langle m^0, X[\vartheta,t_*]x[t_*]\rangle +$$
$$+\sigma(x[t_*^0[\cdot]t_*]) + \int_{t_*}^{t^*}\langle m^0, X[\vartheta,\tau](B(\tau)u^0[\tau] + C(\tau)v_*[\tau])\rangle d\tau + \varphi_2^{(4)'}(t_*,m^0,\nu^0) \qquad (5.34)$$

On the other hand, since $(m^0, \nu^0) \in G_1^{(4)}(t_*)$, it follows from (5.15), as the upper convex hull $\varphi_1^{(4)}(\,\cdot\,)$ is a majorant for the function $\psi_1^{(4)}(\,\cdot\,)$ (5.13), that

$$e_{(4)}(x[t_*^0[\cdot]t_*],\Delta_k) \geq \sigma(x[t_*^0[\cdot]t_*]) + \varkappa(x[t_*^0[\cdot]t_*])(1-\nu^0) +$$
$$+\langle m^0, X[\vartheta,t_*]x[t_*]\rangle + \Delta\psi_1(t_*,m^0) + \varphi_2^{(4)'}(t_*,m^0,\nu^0) \qquad (5.35)$$

The truth of Theorem 5.1 now follows from (5.33)–(5.35), taking (5.5) ($j = 1$) and (5.18) into consideration.

Theorem 5.2 (ν-stability of $e_{(4)}(\,\cdot\,)$). Suppose that the history $x[t_*^0[\,\cdot\,]t_*]$, $t_*^0 \leq t_* < \vartheta$ of a motion of system (5.1) has been realized and that a partition $\Delta_k\{\tau_j\}$ (5.4) of the interval $[t_*, \vartheta]$ has been prescribed.

Then, for any admissible realization $u_*[t_*[\cdot]t^*$, where $t^* = \tau_2 \in \Delta_k\{\tau_j\}$, an admissible realization $v[t_*[\cdot]t^*)$, exists such that

$$e_{(4)}(x[t_*^0[\cdot]t^*],\Delta^*_{k^*}) - e_{(4)}(x[t_*^0[\cdot]t_*],\Delta_k) \geq 0 \tag{5.36}$$

Proof. By (5.13), (5.15) and by Carathéodory's theorem [13, p. 155], given the upper convex hill $\varphi(\cdot)$, under the present conditions a pair $(m_0, \nu_0) \in G_1^{(4)}(t_*)$ exists such that

$$e_{(4)}(x[t_*^0[\cdot]t_*],\Delta_k) = \sigma(x[t_*^0[\cdot]t_*]) + \varkappa(x[t_*^0[\cdot]t_*])(1-\nu_0) +$$
$$+\langle m_0, X[\vartheta,t_*]x[t_*]\rangle + \Delta\psi_1(t_*,m_0) + \varphi_2^{(4)'}(t_*,m_0,\nu_0) \tag{5.37}$$

Now, as in the proof of Theorem 5.1, we consider the auxiliary quantity $e'_{(4)}(x[t_*^0[\cdot]t_*], t^*, x[t^*], \Delta^*_{k^*})$ (5.17).

Relying on measurable selection theorem [15], let us take an admissible realization $v_0[t_*[\cdot]t^*)$ that satisfies the condition

$$\int_{t_*}^{t^*} \langle m_0, X[\vartheta,\tau]C(\tau)v_0[\tau]\rangle d\tau = \int_{t_*}^{t^*} \max_{v\in Q}\langle m_0, X[\vartheta,\tau]C(\tau)v\rangle d\tau \tag{5.38}$$

Let $x[t_*^0[\cdot]t^*]$ be the history of the motion of system (5.1) realized under the action of the controls $u_*[t_*[\cdot]t^*)$ and $v_0[t_*[\cdot]t^*)$. It then follows from (5.17), by using the Cauchy formula, that

$$e'_{(4)}(x[t_*^0[\cdot]t_*],t^*,x[t^*],\Delta^*_{k^*}) \geq \varkappa(x[t_*^0[\cdot]t_*])(1-\nu_0) + \langle m_0, X[\vartheta,t_*]x[t_*]\rangle +$$
$$+\sigma(x[t_*^0[\cdot]t_*]) + \int_{t_*}^{t^*}\langle m_0, X[\vartheta,\tau]B(\tau)u_*[\tau] + C(\tau)v_0[\tau])\rangle d\tau + \varphi_2^{(4)'}(t_*,m_0,\nu_0) \tag{5.39}$$

It now follows from (5.37) and (5.39), in accordance with (5.5) ($j = 1$) and (5.38) and in view of (5.18) (which is proved in exactly the same way as in the proof of Theorem 5.1), that inequality (5.36) is true. This completes the proof of Theorem 5.2.

If we note that, by (5.6)

$$e_{(4)}(x[t_*^0[\cdot]\vartheta],\Delta_k) = \gamma_{(4)}(x[t_*^0[\cdot]\vartheta])$$

we obtain the following proposition as a corollary of Theorems 5.1 and 5.2.

Theorem 5.3. For any history $x[t_*^0[\cdot\,|\,t_*]$, $t_*^0 \leq t_* < \vartheta$ of the motion of system (5.1) and any sequence of partitions $\Delta_k\{\tau_j\}$ (5.4) of the mesh $\delta_k = \max_j(\tau_{j+1} - \tau_j)$ ($k = 1, 2, \ldots$) such that $\lim \delta_k = 0$, $k \to \infty$, the following equality holds

$$\lim_{k\to\infty} e_{(4)}(x[t_*^0[\cdot]t_*],\Delta_k) = \rho^0_{(4)}(x[t_*^0[\cdot]t_*])$$

where $\rho^0_{(4)}(x[t_*^0[\cdot]t_*])$ is the value of game (5.1)–(5.3).

We have thus established that the procedure described above for calculating $e_{(4)}(\cdot)$ on the basis of the functions $\varphi_j^{(4)}(\cdot)$, which are obtained by convex closure of the functions $\psi_j^{(4)}(\cdot)$ in domains $G_j^{(4)}$ with respect to the pair of arguments (m, ν), produces the value $\rho_0^{(4)}(\cdot)$ of game (5.1)–(5.3).

The example in the next section will show that the convex closure must be carried out with respect to the pair (m, ν). In that example, convex closure with respect to m alone for each fixed ν will not yield the value of the game.

6. EXAMPLE

Consider the following problem of type (5.1)–(5.3).

Let the system be described by the equation

$$dx/dt = f(t)u + g(t)v, \quad t_*^0 = 0 \leq t \leq \vartheta = 3 \tag{6.1}$$
$$x = (x_1, x_2) \in R^2, \quad u = (u_1, u_2) \in R^2, \quad v = (v_1, v_2) \in R^2$$

where u and v obey constraints

$$u_1^2+u_2^2\leqslant 1,\quad v_1^2+v_2^2\leqslant 1 \tag{6.2}$$

and $f(t)$ and $g(t)$ are scalar functions

$$f(t)=\begin{cases}2, & 0\leqslant t<1\\ 1, & 1\leqslant t\leqslant 3\end{cases},\quad g(t)=\begin{cases}1, & 0\leqslant t<1\\ 2, & 1\leqslant t\leqslant 3\end{cases} \tag{6.3}$$

The performance index is given as a functional

$$\gamma=\gamma(x[0[\cdot]3])=\max\{|x_2[0]|,|x_2[2]|\}+|x_1[3]| \tag{6.4}$$

where $|\cdot|$ denotes the modulus of the scalar quantity, $N_1=1$, $t_1^{[1]}=\vartheta=3$, $D_1^{(1)}=[1\ 0]$, $N_2=2$, $t_2^{[1]}=t_*^0=0$, $t_2^{[2]}=2$; $D_2^{[1]}=D_2^{[2]}=[0\ 1]$.

Let $\varphi_j(t_*, m, \nu)$ be upper convex hulls with respect to (m, ν) constructed for problem (6.1)–(6.4) in accordance with the procedure (5.4)–(5.13); let $e(\cdot, \Delta_k)$ denote the quantity (5.15) obtained in this case, and let $\rho(\cdot)$ be the value of game (6.1)–(6.4).

Let $\varphi^*_j(t_*, m, \nu)$ denote the functions obtained, using (6.1)–(6.4), by a procedure similar to (5.4)–(5.13) but taking upper convex hulls with respect to m only, for each fixed $\nu\in[0, 1]$; let $e^*(\cdot, \Delta_k)$ denote the quantity calculated like (5.15) from the functions $\varphi_j(t_*, m, \nu)$, and let $\rho^*(\cdot)$ be the limit of $e^*(\cdot, \Delta_k)$ $(k\to\infty)$ as the mesh of the partition Δ_k tends to zero.

Let $t^*=t_*^0=0$. Prescribe a partition Δ_k $\{\tau_j\}$ of the time interval $[0, 3]$ which includes the times 0, 1, 2, 3.

Carrying out the calculations, we obtain

$$G_1(0)=\{(m=(m_1,m_2),\nu):\ 0\leqslant\nu\leqslant 1,\ |m_1|\leqslant 1,\ |m_2|\leqslant\nu\} \tag{6.5}$$

$$\varphi_1(0,m,\nu)=-\sqrt{m_1^2+m_2^2}+(1+(\sqrt{2}-1)\nu)+1$$

$$\varphi_1^*(0,m,\nu)=-\sqrt{m_1^2+m_2^2}+\sqrt{1+\nu^2}+1,\quad (m,\nu)\in G_1(0)$$

It can be seen that the domain $G_1(0)$ does not have the homogeneity property (4.5), and the function $\varphi^*(0, m, \nu)$ is neither concave nor homogeneous with respect to the pair (m, ν).

Note that in this problem the limits $\rho(\cdot)$ and $\rho^*(\cdot)$ can be calculated fairly easily by an analytical procedure in terms of the quantities $e(\cdot, \Delta_k)$ and $e^*(\cdot, \Delta_k)$. This may be seen from (6.5).

Let $x_1[t_*^0]=2$, $x_2[t_*^0]=2$ be a given initial state.

Then, using (6.5), we compute

$$\rho(x[0[\cdot]0]=(2,2))=6-\sqrt{2(\sqrt{2}-1)}\approx 5{,}089>\rho^*(x[0[\cdot]0])=5 \tag{6.6}$$

The strict inequality (6.6) shows that $\rho^*(\cdot)$ is not the value of game (6.1)–(6.4).

The control process generated by the constructions described above was modelled on a computer for problem (6.1)–(6.4). The results of the numerical experiment corroborate our theoretical conclusions. For example, accurate simulation of the control process for optimum strategies $u^0(\cdot)$ and $v^0(\cdot)$ that are extremum strategies for $\rho(\cdot)$ gave $\gamma=5.089=\rho(x[0[\cdot]0])$. In the case of the optimum strategy $u^0(\cdot)$ and the extremum strategy $v^*(\cdot)$ for $\rho^*(\cdot)$, the result was $\gamma=5.28>\rho(x[0[\cdot]0])$.

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